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Nonnegative Quadratic Forms and Bounds on Orthogonal Polynomials

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We show that some nonnegative quadratic forms containing orthogonal polynomials, such as e.g. the Christoffel–Darboux kernel for $x=y$ in the classical case, provide a lot of information about behavior of the polynomials on the real axis. We illustrate the method for the case of Hermite polynomials and use it to derive new explicit bounds for binary Krawtchouk polynomials. © 2001 Academic Press

Key Words: orthogonal polynomials; Hermite polynomials; Krawtchouk polynomials; bounds.

1. INTRODUCTION

The aim of this paper is to describe an elementary method for obtaining explicit bounds on orthogonal polynomials on the real axis and their extreme zeros. We will illustrate it for the case of Hermite polynomials and use this approach to derive new bounds on the binary Krawtchouk polynomials.

The basic idea of the method is to study certain nonnegative quadratic forms containing orthogonal polynomials. One of the examples of such a form in the classical case is the Christoffel–Darboux formula for $x=y$ (we will use a term Christoffel–Darboux kernel for this special case),

$$W_k(x)=h_k\sum_{i=0}^{k}\frac{f_i^2(x)}{h_i}=f'_{k+1}(x)\,f_k(x)-f'_k(x)\,f_{k+1}(x)\geqslant 0, \tag{1}$$

where $\{f_i(x)\}$ is a family of monic orthogonal polynomials, and $\|f_k\|_{L_2}=\sqrt{h_k}$.

For the classical orthogonal polynomials the derivatives here can be expressed as a linear combination of $f_k(x)$ and $f_{k+1}(x)$. So the right hand side of (1) becomes a quadratic form $Q_k(v,u):=Q(v,u)$ in variables $v=f_k(x)$ and $u=f_{k+1}(x)$, with coefficients depending on x and k. It turns out that $Q(v,u)$ is an indefinite quadratic form, whenever as a function in

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x it is clearly nonnegative. For instance, it will be shown with Hermite polynomials that $W_k(x)=Q_k(v,u)=u^2-2xvu+2(k+1)\,v^2$. Given such a form one can extract a lot of information about the behaviour of the polynomials in a pure algebraic manner. Observe that positivity of $Q(v,u)$ implies some inequalities on v/u, provided the discriminant Δ of the form is nonnegative. Since in the oscillatory region v/u may have any real value, it follows that there are no roots of u in the domain $\{x\colon \Delta\geqslant 0\}$. In practice, if the coefficients of the form depend on x, some calculations may be needed to justify this conclusion. As well the inequalities on v/u for $\Delta\geqslant 0$, yield explicit bounds in the monotonic region. Moreover such forms readily give two-sided bounds on the polynomials in the oscillatory region. Indeed, consider the following expression,

$$D_k(v,u)=\frac{d}{dx}\,Q(v,u)-g(x)\,Q(v,u), \tag{2}$$

and rewrite it as a quadratic form in variables v and u. If $g(x)$ is chosen so that $D_k(v,u)$ is positive semidefinite then, by $W_k(x)>0$, we get $W'_k(x)/W_k(x)\geqslant g(x)$. Integrating this form x_1 to x_2 we will obtain

$$\frac{W_k(x_2)}{W_k(x_1)}\geqslant \exp\left(\int_{x1}^{x2} g(x)\,dx\right),$$

yielding a sought lower bound. Furthermore, if there is an appropriate x_1 such that one can calculate or estimate $W_k(x_1)$ this gives a bound on $W_k(x)$ as well. Similarly, for an upper bound $D_k(v,u)$ should be chosen negative semidefinite. Having an upper bound on $W_k(x)$ one readily gets an upper bound on $f_k(x)$, e.g. as the major axes of ellipse $Q(v,u)$. These bounds make no sense near to the ends of the interval, but considering in the similar manner another form, e.g. $Q(v,u)+v^2$, one gets an upper estimate for all x. To establish semi-definiteness of $D_k(v,u)$ it is enough to choose $g(x)$ so that the discriminant of $D_k(v,u)$ vanishes. Then $g(x)$ is just the solutions of a quadratic equation, and two its roots give a lower and an upper bound.

For classical polynomials of a discrete variable, whenever the minimum distance between the roots is greater than $\sqrt{2}$, one can use $f^2(x)-f(x+1)\,f(x-1)>0$, (Theorem 7 below). Since Krawtchouk, Meikxner and Charlier polynomials are self dual (i.e. one can interchange k and x) we can convert this into a quadratic form via the three-term recurrence. As well, one can replace (2) by its difference analogue $Q_{k+1}(v,u)-g(x)\,Q_k(v,u)$ and use the three-term recurrence. This yields a different type of bounds, less informative but giving, in a sense, dual estimates on the form.

The sharpness of the results obtained by such a method depends, of course on the chosen form. For Hermite case, based on a higher degree generalization of Laguerre inequality $(f')^2 - ff'' \geqslant 0$, bounds with the explicit error term of order k^{-2} has been given in [7], (see Theorem 6 below).

Besides the Christoffel–Darboux kernel there are so-called the Turan type inequalities written under a suitable normalization as $f_k^2(x) - f_{k-1}(x)\, f_{k+1}(x) \geqslant 0$, also giving the sought quadratic form (see [9, 12, 26]). Another source of quadratic forms, useful when we have a differential equation of the second order, is the above Laguerre and Love $(k-1)(f')^2 - kff'' \geqslant 0$, inequalities where f is a polynomial of degree k having only real zeros [20]. Also one can appeal to the following theorem of Newton (see e.g. [1] and also [2] for generalization on entire functions):

THEOREM 1. *If all the roots of* $P(x) = \sum_{i=0}^{n} a_i \binom{n}{i} z^i$, $n \geqslant 2$, *are real, then* $a_i^2 \geqslant a_{i-1} a_{i+1}$ *for* $i = 1, ..., n-1$, *and the inequality is strict unless all the roots are equal.*

For instance, Krawtchouk polynomials $K_k(x; n, q) := K_k(x)$, are defined by the following generating function:

$$\sum_{k=0}^{n} K_k(x)\, z^k = (1-z)^x\, (1 + (q-1)\, z)^{n-x}. \tag{3}$$

Applying the above theorem we get for nonnegative integers x and $k = 1, ..., n-1$,

$$(K_k(x))^2 - \frac{(k+1)(n-k+1)}{k(n-k)}\, K_{k-1}(x)\, K_{k+1}(x) \geqslant 0, \tag{4}$$

and inequality is strict, provided $q \neq 0$ and $x \neq 0, n$.

For Hermite polynomials the identity ([24], p. 640)

$$\sum_{i=0}^{n} \frac{t^k H_{2k+\delta}(x)}{(n-k)!\,(2k+\delta)!} = \frac{(t-1)^n}{(2n+\delta)!} \left(\frac{t-1}{t}\right)^{\delta/2} H_{2n+\delta}\left(x \sqrt{\frac{t-1}{t}}\right), \qquad \delta = 0, 1;$$

gives the following form

$$(H_{2k+\delta}(x))^2 - \frac{2k+2\delta-1}{2k+2\delta+1}\, H_{2k+\delta+2}(x)\, H_{2k+\delta-2}(x) \geqslant 0, \tag{5}$$

which is not proportional to the Christoffel–Darboux kernel. A few more such sums for the classical polynomials may be found in [24].

The paper is organized as follows: We start with the Hermite polynomials to illustrate the method and to reveal its scope on the simplest example. Since calculations are routine whatever quadratic form is used, we will derive estimates capturing only the main term of the corresponding asymptotics. A much more precise result available at the expense of more extensive calculations [6, 7] are stated at the end of the section. The last section deals with the binary Krawtchouk polynomials ($q=2$). We show how the method can be adjusted for the discrete case and use it to derive new bounds for Krawtchouk polynomials on the whole real axis. We also refine here asymptotics for monotonic region given in [11] establishing, in particular, an estimate for the error term.

2. HERMITE POLYNOMIALS

The Hermite polynomials are defined by $H_0(x)=1$, $H_1(x)=2x$, and the recurrence $H_{k+1}(x)=2xH_k(x)-2kH_{k-1}(x)$. We also need $H'_k(x)=2kH_{k-1}(x)$.

We start with bounds on the Christoffel–Darboux kernel $W_k(x)$ in the oscillatory region. Put throughout this section $v=H_{k-1}(x)$, $u=H_k(x)$, $w=H_{k+1}(x)$.

THEOREM 2. *For* $|x|\leqslant\sqrt{2k+2}$,

$$1-\frac{|x|}{\sqrt{2k+1}}\leqslant\frac{\left(\frac{k}{2}\right)!^2}{2(k+1)!\,k}\,W_k(x)\,e^{-x^2}\leqslant 1+\frac{|x|}{\sqrt{2k+2}},\qquad k \text{ even};$$

$$1-\frac{|x|}{\sqrt{2k+2}}\leqslant\frac{\left(\frac{k+1}{2}\right)!^2}{(k+1)!^2}\,W_k(x)\,e^{-x^2}\leqslant 1+\frac{|x|}{\sqrt{2k+2}},\qquad k \text{ odd}.$$

Proof. Rewrite $W_k(x)$ and its derivative in x as quadratic forms in v and u:

$$\begin{aligned} Q_k(v,u) &:= W_{k-1}(x)=2^k(k-1)!\sum_{i=0}^{k-1}\frac{(H_i(x))^2}{2^i i!}\\ &= u^2-2xvu+2kv^2, \end{aligned}\tag{6}$$

$$\frac{d}{dx}Q_k(v,u) = W'_{k-1}(x)=2xu^2-2(2x^2+1)\,vu+4xkv^2.$$

It is enough to consider $x \geqslant 0$. The discriminant of $\frac{d}{dx} Q_k(v, u) - g(x)\, Q_k(v, u)$ vanishes for

$$g_1(x) = 2x + \frac{1}{\sqrt{2k+x}}, \qquad g_2(x) = 2x - \frac{1}{\sqrt{2k-x}}.$$

We obtain

$$W'_{k-1}(x) - g_1(x)\, W_{k-1}(x) = -\frac{(u + v\sqrt{2k})^2}{\sqrt{2k+x}} \leqslant 0,$$

$$W'_{k-1}(x) - g_2(x)\, W_{k-1}(x) = \frac{(u - v\sqrt{2k})^2}{\sqrt{2k-x}} \geqslant 0, \qquad x < \sqrt{2k}.$$

and thus, $g_2(x) \leqslant W'_{k-1}(x)/W_{k-1}(x) \leqslant g_1(x)$. Integrating this from 0 to x we get

$$\frac{\sqrt{2k}-x}{\sqrt{2k}}\, e^{x^2} W_{k-1}(0) \leqslant W_{k-1}(x) \leqslant \frac{\sqrt{2k}+x}{\sqrt{2k}}\, e^{x^2} W_{k-1}(0).$$

The result follows by $W_k(0) = k!^2/(\frac{k}{2})!^2$, for k even and $2k!(k-1)!/(\frac{k-1}{2})!^2$, for k odd. ∎

It is well-known that the largest root of $H_k(x)$ is $\sqrt{2k} + O(k^{-1/6})$ [27]. Putting $|x| = \lambda \sqrt{2k+2}$, $0 < \lambda < 1$, we find then the ratio between the lower and the upper bound in the theorem is $\frac{1-\lambda}{1+\lambda}$. This reflects an important feature in the behaviour of many of orthogonal polynomials, that $w(x)\, f^2(x)$ is almost independent on x in the oscillatory region. We refer to [18] for interesting discussion.

Let us see what can be done without using derivatives of $H_k(x)$.

THEOREM 3. *For x fixed, a function $W_k(x)/2^k k!$ increases in k, whenever for $|x| < \sqrt{2k+2}$, $W_k(x)/2^k k!\,(2k+2-x^2)$ decreases in k.*

Proof.

$$W_k(x) - 2k W_{k-1}(x) = 2u^2 \geqslant 0,$$

$$W_k(x) - \frac{2k(2k+2-x^2)}{2k-x^2}\, W_{k-1}(x) = \frac{(2xu-2kv)^2}{x^2-2k}.$$

From the first expression we get $\prod_{i=k+1}^{l} W_i(x) > \prod_{i=k+1}^{l} 2i W_{i-1}(x)$, hence $W_l(x) \geqslant 2^{l-k} l!/k!\; W_k(x)$, proving the first claim. Multiplying the second inequality, we complete the proof. ∎

In a sense, these estimates are dual (and weaker) to the obtained early. Indeed, by $W_0(x)=2$ it follows $W_k(x)>2^{k+1}k!$. Thus, here we have an unconditional lower bound, whenever the upper bound holds only inside the oscillatory region.

Now we fined bounds on the extreme roots of H_k and an inequality in monotonic region. To get slightly sharper estimates consider instead of $W_k(x)$ the following form

$$Q_k(v, u)=u^2-2xvu+2(k-1)\,v^2\geqslant 0. \tag{7}$$

The inequality here holds by $Q_1(v, u)=0$ and $Q_{k+1}(u, w)-2kQ_k(v, u)=4kv^2\geqslant 0$.

THEOREM 4. *All the roots of $H_k(x)$, $k>1$, are in the interval* $(-\sqrt{2k-2}, \sqrt{2k-2})$. *Moreover, for* $x\geqslant\sqrt{2k-2}$,

$$H_{k-1}(x)<\frac{x-\sqrt{x^2-2k+2}}{2k-2}\,H_k(x).$$

Proof. Since $u^2-2xvu+2(k-1)\,v^2>0$ for $k>1$, then for $x\geqslant\sqrt{2k-2}$,

$$r(x)=\frac{v}{u}\notin J=\left[\frac{x-\sqrt{x^2-2k+2}}{2k-2}, \frac{x+\sqrt{x^2-2k+2}}{2k-2}\right].$$

Let ν_k be the largest root of $H_k(x)$. Then $r(x)$ monotonically decreases from ∞ to 0 on (ν_k, ∞). Thus $\nu_k<\sqrt{2k-2}$, since otherwise $r(x)$ intersects J. Moreover, since $\lim_{x\to\infty} r(x)=0$, whenever $\lim_{x\to\infty}(x+\sqrt{x^2-2k+2})/(2k-2)=\infty$, the only possibility is $r(x)<(x-\sqrt{x^2-2k/2})/(2k-2)$. ∎

It is worth also noticing that using form (5) one obtains also bounds for the least positive root of $H_k(x)$. Namely, it gives that all positive roots of $H_{2k+\delta}(x)$, $\delta=0, 1$, are in

$$\left(\sqrt{2k-\tfrac{5}{2}-\sqrt{(2k-1-\delta)(2k-4+\delta)}},\right.$$
$$\left.\sqrt{2k-\tfrac{5}{2}+\sqrt{(2k-1-\delta)(2k-4+\delta)}}\right).$$

we omit the details.

Having estimates for a quadratic form, one can readily find an upper estimate for $|H_k(x)|$.

THEOREM 5. *For* $k > 1$,

$$(H_k(x))^2 \leqslant (2k+1)\, k!^2 \left(\frac{k}{2}\right)!^{-2} e^{x^2}, \qquad k \text{ even},$$

$$(H_k(x))^2 \leqslant (k+1)!^2 \left(\frac{k+1}{2}\right)!^{-2} e^{x^2}, \qquad k \text{ odd}.$$

For $|x| < \sqrt{2k+1}$ *and* $k > 1$,

$$(H_k(x))^2 \leqslant \frac{(2k+1)\, k!^2}{(2k+1-x^2)\left(\frac{k}{2}\right)!^2} e^{x^2}, \qquad k \text{ even},$$

$$(H_k(x))^2 \leqslant \frac{(k+1)!^2}{(2k+1-x^2)\left(\frac{k+1}{2}\right)!^2} e^{x^2} \qquad k \text{ odd}.$$

Proof. By (7),

$$v^2 \leqslant v^2 + u^2 - 2xvu + (2k-2)\, v^2 = u^2 - 2xvu + (2k-1)\, v^2 = Q_k(v, u).$$

Denote by R_k the value of $Q_k(v, u)$ for $x = 0$,

$$R_k = k! \binom{k}{k/2}, \quad k \text{ even}; \qquad R_k = (2k-1)(k-1)! \binom{k-1}{(k-1)/2}, \quad k \text{ odd}.$$

To estimate $Q_k(v, u)$ one just repeats the arguments of Theorem 2 considering $\frac{d}{dx} Q_k(v, u) - g(x)\, Q_k(v, u)$, for $g(x) = 2x$. This gives $Q_k(v, u) \leqslant R_k e^{x^2}$, proving the first two inequalities. The second two follow from

$$(2k-1-x^2)\, v^2 \leqslant (2k-1-x^2)\, v^2 + (u - xv)^2 = Q_k(v, u). \quad \blacksquare$$

The results of the two last theorem are well known and relatively weak, see e.g. [27]. Yet our approach applied to a more complicated quadratic form than those above, yields the following estimate [7] with the error term of order k^{-2} for x not too close to $\sqrt{2k}$.

THEOREM 6. *For* $x^2 < 2k - \frac{3}{2}$

$$(H_k(x))^2\, e^{-x^2} \leqslant C(k) \frac{2y^2 - 4y + 3}{\sqrt{y(4y^4 - 12y^3 + 9y^2 + 10ky - 12k)}} \times \exp\left(\frac{15x^2}{2y(2y-3)^2}\right),$$

where $y = 2k - x^2$,

$$C(k) = \frac{2k\sqrt{4k-2}\,k!^2}{\sqrt{8k^2-8k+2}\,(k/2)!^2}, \qquad \textit{for } k \textit{ even},$$

$$C(k) = \frac{\sqrt{16k^2-16k+6}\,k!\,(k-1)!}{\sqrt{2k-1}\,((k-1)/2)!^2}, \qquad \textit{for } k \textit{ odd}.$$

The bound is sharp in a sense that replacing $\exp(15x^2/(2y(2y-3)^2))$ by $\exp(-15x^2/(2y(2y-3)^2))$ the inequality reverses at all roots of the equation

$$xy(2y-3)\,H_k(x) = (2y^2-4y+3)\,H_{k-1}(x).$$

As well, for the largest root the method gives upper bound [6]

$$\sqrt{(4k-3k^{1/3}-1)/2} = \sqrt{2k} - \frac{3}{4\sqrt{2}}k^{-1/6} + O(k^{-1/2}).$$

It differs only by a weaker constant at $k^{-1/6}$ term from the true asymptotics [27].

3. KRAWTCHOUK POLYNOMIALS

Although bounds for the Krawtchouk polynomials play a crucial role in so-called linear-programming approach in the coding theory [17, 22], very few is known, especially in comparing with the classical case. It seems, many analogies of the well-known properties of the classical polynomials, such as e.g. monotonicity of the consecutive maxima, or of the distances between consecutive zeros, which certainly true for the Krawtchouk polynomials have never been established. Concerning the bounds, asymptotics outside the interval containing the roots has been obtained in [11] for k growing linearly with n. A strong asymptotic based on the generating function (3) has been given in [10]. Recently a more general result under weaker assumptions was also proved [19]. The asymptotics of zeros has been obtained via potential theory approach by Dragnev and Saff [4, 5] (see also [15, 16]). However for many applications it is more convenient to have explicit bounds rather than asymptotics with hard to estimate error terms. In this direction, for n and k even, Solé [25] (see also [13]), gave a simple inequality

$$|K_k(x)| \leqslant \binom{n}{n/2}\binom{n/2}{k/2}\binom{n}{x}^{-1}.$$

Explicit bounds for Christoffel–Darboux kernel has been recently given in [14].

Here, based on the quadratic form (4) we will establish some explicit bounds on the binary Krawtchouk polynomials. Although these results captured only the main term of the corresponding asymptotics, probably much more precise bounds may be obtained by a similar technique at the expanse of more involved calculations, see e.g. [6].

We start with listing some properties of the Krawtchouk polynomials. Proofs can be found in [22]. For the sake of simplicity we will deal here only with the binary case, that is when the Krawtchouk polynomials are defined by

$$\sum_{i=0}^{\infty} K_i(x)\, z^i = (1-z)^x\, (1+z)^{n-x}. \tag{8}$$

We need the following recurrences:

$$(k+1)\, K_{k+1}(x) = (n-2x)\, K_k(x) - (n-k+1)\, K_{k-1}(x), \tag{9}$$

$$(n-x)\, K_k(x+1) = (n-2k)\, K_k(x) - xK_k(x-1). \tag{10}$$

Krawtchouk polynomials possess some important symmetry properties (notice that similar transformations also hold for other classical orthogonal polynomials of discrete variable)

$$K_k(n-x) = (-1)^k\, K_k(x), \tag{11}$$

and for integer $x \in [0, n]$,

$$\binom{n}{x} K_k(x) = \binom{n}{k} K_x(k), \tag{12}$$

$$K_k(x) = (-1)^x\, K_{n-k}(x). \tag{13}$$

For x beyond the interval $[0, n]$ $K_k(x)$ can be easily estimated by the explicit formula

$$K_k(x) = \sum_{j=0}^{k} (-1)^j \binom{x}{j} \binom{n-x}{k-j},$$

which contains only positive terms. Thus, without loss of generality we can deal with $x \leqslant n/2$ and, as far as we consider only integer values of x in $[0, n]$, with $k \leqslant n/2$.

We will need also some particular values

$$K_k(0)=\binom{n}{k}, \qquad K_k(1)=\frac{n-2k}{n}\binom{n}{k},$$
$$K_k(n/2)=0, \quad k \text{ odd}; \qquad K_n(n/2)=(-1)^{k/2}\binom{n/2}{k/2}, \quad n \text{ even}. \tag{14}$$

It will be convenient to use the following notation:

$$y=2\sqrt{x(n-x)}, \qquad p=2\sqrt{k(n-k)}, \qquad s=\sqrt{(n-2k)^2-4x(n-x)}.$$

The Krawtchouk polynomials are orthogonal on $[0, n]$, namely

$$\sum_{x=0}^{n}\binom{n}{x} K_k(x)\, K_i(x)=\delta_{ki}h_k=2^n\binom{n}{k}\delta_{ki}. \tag{15}$$

Thus, all zeros of $K_k(x)$ are simple and lie in $[0, n]$. It is worth noticing that for orthogonal polynomials of discrete variable an orthogonality relation readily gives an upper bound for integer values of x from the interval of orthogonality. For, it is enough to replace the sum by a single term. In our case this yields for $0 \leqslant x \leqslant n$,

$$(K_k(x))^2<2^n\binom{n}{k}\binom{n}{x}^{-1}. \tag{16}$$

Surprisingly, as we will show, this trivial bound is of the right order of magnitude for the oscillatory region, $x \in [\frac{n-p}{2}, \frac{n+p}{2}]$.

Besides quadratic form (4) there is another more convenient nonnegative form in variables $v=K_k(x)$, $u=K_k(x+1)$, also giving a closer analogue of differential inequalities for the classical polynomials. Namely, from (4) and (12) we conclude that the following quadratic form is nonnegative for integer $x \in [0, n]$,

$$U_k(x)=(K_k(x))^2-K_k(x+1)\, K_k(x-1)$$
$$=\frac{1}{x}(xv^2-(n-2k)\, vu+(n-x)\, u^2) \geqslant 0, \tag{17}$$

and is strictly positive for $x=1, 2, ..., n$, where $v=K_k(x)$, $u=K_k(x+1)$.

First, let us get rid of the restrictions on x.

Let $v_1<\cdots<v_k$ be the roots of $K_k(x)$. The following lemma has been proved in [3]:

LEMMA 1. *Let $k < n/2$, then $v_{i+1} - v_i > 2$.*

Notice also that by (13) the roots of $K_{n/2}(x)$ are just the all odd integers of $[0, n]$.

THEOREM 7. *Let $f(x)$ be a polynomial having only distinct real roots $x_1 < \cdots < x_k$, such that $x_{i+1} - x_i \geqslant \sqrt{2}$ for $i = 1, ..., k-1$. Then $f^2(x) - f(x-1)\, f(x+1) \geqslant 0$.*

In particular

$$U_k(x) = (K_k(x))^2 - K_k(x+1)\, K_k(x-1) > 0,$$

provided $1 \leqslant k \leqslant n/2$.

Proof. Observe that for $x = v_i$, we have $f(x)^2 - f(x-1)\, f(x+1) > 0$ by simplicity of zeros. So we may assume $x \neq v_i$. Put $f(x) = a \prod_{i=0}^{k} (x - x_i)$, and $d_i = 1 - \frac{1}{(x-x_i)^2}$. It is enough to show that $F(x) = (f(x-1)\, f(x+1))/f^2(x) = \prod_{i=0}^{k} d_i < 1$. Note that $|x - x_i| > \sqrt{2}$ implies $0 < d_i < 1$. Hence there are at most two d_i such that $|d_i| > 1$. Moreover, since $|d_i| > 1$ only if $d_i < 0$ then for $x < x_1$ or $x > x_k$ we get that either all d_i are less than one or $F(x) < 0$. Thus we may assume that there is an index j such that $x_j < x < x_{j+1}$. By $x_{i+1} - x_i > \sqrt{2}$ we have $0 < d_i < 1$, $i \neq j, j+1$. Now the following inequalities complete the proof:

$$d_j d_{j+1} \leqslant \max_{x_j < x < x_{j+1}} d_j d_{j+1} = \left(1 - \frac{4}{(x_{j+1} - x_j)^2}\right)^2 < 1. \quad \blacksquare$$

Solving quadratic Eq. (17) for v/u yields the following bounds

THEOREM 8. *All roots of $K_k(x)$, $k < n/2$, are in $(\frac{n-p}{2} + 1, \frac{n+p}{2} - 1)$, and for $0 < x < n$ beyond this interval*

$$\frac{K_k(x+1)}{K_k(x)} > \frac{n - 2k + s}{2(n-x)}.$$

The bounds for the extreme roots can be improved using a more complicated form. For the case $k/n \to c$, $0 < c < \frac{1}{2}$, the required calculations has been done in [6] giving for the least root

$$v_1 \geqslant \frac{n-p}{2} + \frac{3}{8}(n-2k)^{2/3} \left(\frac{p}{2}\right)^{-1/6} + O(1).$$

We believe that up to a multiplicative constant the extra term here is of the right order of magnitude. The best currently available upper bound is due to Levenshtein (see [17]),

$$v_1 \leqslant \frac{n-p}{2} + k^{1/6}\sqrt{n-k}, \qquad k \leqslant \frac{n}{2}.$$

The inequality on $K_k(x)$ of the above theorem actually is tight, and have been used in [11] to given an asymptotics for Krawtchouk polynomials in monotonic region for k growing linearly with n. We will show that this result holds under much weaker conditions and give an estimate for the error term. The following lemma is a refinement of arguments presented in [11].

LEMMA 2. *Let $k < n/2$ and $x < \frac{n-p}{2}$, then for some $0 < \theta < 1$,*

$$\frac{K_k(x+1)}{K_k(x)} = I(x) + \theta R(x), \tag{18}$$

where

$$I(x) = \frac{n-2k+s}{2(n-x)}, \qquad R(x) = \frac{x(n-2x+p)}{s^3}.$$

Proof. Let $v = v_1$ be the least root of $K_k(x)$ and let $\delta = v - x$. By the assumption and Theorem 8 we have $\delta > 1$. Let $r(x) = K_k(x+1)/K_k(x)$, and put $\mu_0 = e^{1/(4\delta-4)}$. First we will show

$$1 < \frac{r(x-1)}{r(x)} < \mu_0^2. \tag{19}$$

We have

$$\frac{r(x-1)}{r(x)} = \frac{(K_k(x))^2}{K_k(x-1)\,K_k(x+1)} = \prod_{i=1}^{k} \frac{(v_i - x)^2}{(v_i-x)^2 - 1} > 1.$$

On the other hand, by Lemma 1, $(v_i - x)^2 > (\delta + 2i)^2$. Taking logarithm of the above product and using $\ln(1+z) < z$, we obtain

$$\ln \frac{r(x-1)}{r(x)} < \sum_{i=0}^{k-1} \frac{1}{(v_i-x)^2-1} < \frac{1}{2}\sum_{i=0}^{k-1}\left(\frac{1}{\delta+2i-1} - \frac{1}{\delta+2i+1}\right)$$
$$< \frac{1}{2\delta-2},$$

and (19) follows. Now, dividing (10) by $K_k(x)$ we get $(n-x)\,r(x) = n-2k+\frac{x}{r(x-1)}$. Hence, by (19), for some $1<\mu<\mu_0$,

$$\mu^2(n-x)\,r^2(x)-\mu^2(n-2k)\,r(x)+x=0.$$

Using $r(0)=\frac{n-2k}{n}$, to select the correct root, we obtain

$$r(x)=\frac{(n-2k)\,\mu+\sqrt{\mu^2(n-2k)^2-4x(n-x)}}{2\mu(n-x)}.$$

To complete the proof, consider the last expression as a function $a(\mu)$ in μ. Set also $b(\mu)=\sqrt{\mu^2(n-2k)^2-4x(n-x)}$, so $b(1)=s$.

Since $a'(\mu)=2x/\mu^2b(\mu)>0$ we get $I(x)=a(1)<r(x)<a(\mu_0)$. For the difference $a(\mu_0)-a(1)$ we obtain:

$$a(\mu_0)-a(1)=\frac{2x(\mu_0^2-1)}{\mu_0(b(\mu_0)+\mu_0 s)}<\frac{2x(\mu_0-1)}{s\mu_0}.$$

since $(e^z-1)/e^x<z$ for $z>0$ the last expression in less than $x/(2\delta-2)\,s$, hence $R(x)<x/(2\delta-2)\,s$. Finally, using $v>\frac{n-p}{2}+1$ we get (18). ∎

We need the values of the following integrals (we used Mathematica for calculations),

LEMMA 3.

$$\int_0^x \ln I(x)\,dx = n\ln\frac{3n-2k-2x-s}{2n} - k\ln\frac{n-2x+s}{2(n-k)}+x\ln\frac{n-2k+s}{2(n-x)}.$$

$$\int_0^x \frac{R(x)}{I(x)}\,dx=\frac{1}{4}\left(\ln\frac{n-p-2x}{n-p}+\frac{(n+p-2x)(n-2k)}{ps}+\frac{n+p}{p}\right).$$

Now we are ready to establish asymptotics on $K_k(x)$ for $x<\frac{n-p}{2}$. For simplicity it will be done only for integer x.

THEOREM 9. *For $k<n/2$ and $0\leqslant x<\frac{n-p-1}{2}$, integer,*

$$\ln K_k(x+1)=\ln\binom{n}{k}+n\ln\frac{3n-2k-2x-s}{2n}+k\ln\frac{n-2x+s}{2(n-k)} +x\ln\frac{n-2k+s}{2(n-x)}+\psi(x),$$

where

$$\ln \frac{n-2k+s}{2(n-x)} < \psi(x) < \frac{4nx(n-x)}{ps(n-2k)}.$$

Proof. Observe that $I(x)$ is a decreasing, while $R(x)$ is an increasing function in x. It is also well-known that for a monotone function $f(x)$,

$$\min(f(0), f(m)) \leqslant \sum_{i=0}^{m} f(i) - \int_0^m f(z)\, dz \leqslant \max(f(0), f(m)). \tag{20}$$

Now, the identity

$$\ln K_k(x+1) = \ln K_k(0) + \sum_{j=0}^{x} \ln \frac{K_k(j+1)}{K_k(j)} = \ln \binom{n}{x} + \sum_{j=0}^{x} \ln r(j),$$

and

$$\sum_{j=0}^{x} \ln r(j) > \ln I(x) + \int_0^x \ln I(x)\, dx,$$

yield the lower bound.

For the upper bound we have

$$\sum_{j=0}^{x} \ln r(j) < \sum_{j=0}^{x} \ln(I(j) + R(j)) < \sum_{j=0}^{x} \left(\ln I(j) + \frac{R(j)}{I(j)} \right)$$

$$< I(0) + \frac{R(x)}{I(x)} + \int_0^x \ln I(x)\, dx + \int_0^x \frac{R(x)}{I(x)}\, dx.$$

By $\ln I(0) < 0$, $\ln \frac{n-p-2x}{n-p} < 0$, using the previous lemma we obtain

$$\frac{R(x)}{I(x)} + \int_0^x \frac{R(x)}{I(x)}\, dx < \frac{(n+p)(s^2+2p)(n-2k-s)}{4ps^3}$$

$$< \frac{2nx(n-x)(s^2+2p)}{p(n-2k)\, s^3}.$$

Since $2p < s^2$ for $x < \frac{n-p-1}{2}$ the result follows. ∎

It is easy to check that for fixed $0 < \varepsilon < 1$, and $0 < \kappa < \frac{1}{2}$, such that $x = \frac{\varepsilon(n-p)}{2}$, $k = \kappa n$, the error term here is $O(1)$.

Now we will find bounds on $U_k(x)$ and $K_k(x)$ using the method applied to Hermite polynomials. For, we shell introduce one more nonnegative (for $x \geqslant 0$) form:

$$V_k(x) = (x+1)(K_k(x))^2 - xK_k(x+1)\, K_k(x-1).$$

Using (10), (11) we get the following particular values:

LEMMA 4.

$$U_k\left(\frac{n}{2}\right) = \frac{p^2}{n^2}\binom{n/2}{k/2}^2,$$

$$V_k\left(\frac{n}{2}\right) = \frac{p^2+2n}{2n}\binom{n/2}{k/2}^2, \qquad n, k \text{ even},$$

$$U_k\left(\frac{n}{2}\right) = 4\binom{n/2-1}{(k-1)/2}^2,$$

$$V_k\left(\frac{n}{2}\right) = 2n\binom{n/2-1}{(k-1)/2}^2 \qquad n \text{ even}, k \text{ odd},$$

$$U_k\left(\frac{n-1}{2}\right) = \frac{4k}{n-1}\binom{(n-1)/2}{k/2}^2,$$

$$V_k\left(\frac{n-1}{2}\right) = (2k+1)\binom{(n-1)/2}{k/2}^2 \qquad n \text{ odd}, k \text{ even},$$

$$U_k\left(\frac{n-1}{2}\right) = \frac{4(n-k)}{n-1}\binom{(n-1)/2}{(k-1)/2}^2,$$

$$V_k\left(\frac{n-1}{2}\right) = (2n-2k+1)\binom{(n-1)/2}{(k-1)/2}^2 \qquad n, k \text{ odd}.$$

THEOREM 10. *Let* $n > 2$ *and* $x \in (\frac{n-p}{2}, \frac{n+p}{2})$ *be an integer. Then for* n *even*:

$$\frac{2x+p-n}{p} \leqslant \frac{\binom{n-2}{x-1} U_k(x)}{\binom{n-2}{n/2-1} U_k\left(\frac{n}{2}\right)} \leqslant \frac{n+p-2x}{p}, \tag{21}$$

For n odd:

$$\frac{2x+p-n}{p-1} \leqslant \frac{\binom{n-2}{x-1} U_k(x)}{\binom{n-2}{(n-3)/2} U_k\left(\frac{n-1}{2}\right)} \leqslant \frac{n+p-2x}{p+1}. \tag{22}$$

For $0 \leqslant x < \lfloor \frac{n}{2} \rfloor$,

$$V_k(x) \leqslant \frac{\binom{n-1}{\lfloor n/2 \rfloor}}{\binom{n-1}{x}} V_k\left(\left\lfloor \frac{n}{2} \right\rfloor\right), \tag{23}$$

Proof. Choose

$$g_1(x) = \frac{x(n+p-2x-2)}{(n-x-1)(n+p-2x)}, \qquad g_2(x) = \frac{x(2x+2+p-n)}{(n-x-1)(2x+p-n)},$$

this yields

$$U_k(x+1) - g_1(x)\, U_k(x) = \frac{(\sqrt{n-p}\, K_k(x+1) - \sqrt{n+p}\, K_k(x))^2}{(n-x-1)(n+p-2x)},$$

$$U_k(x+1) - g_2(x)\, U_k(x) = -\frac{(\sqrt{n+p}\, K_k(x+1) - \sqrt{n-p}\, K_k(x))^2}{(n-x-1)(2x+p-n)}.$$

By $x \in (\frac{n-p}{2}, \frac{n+p}{2})$ the denominators here are positive, thus we get

$$\frac{(n-x-1)(n-p-2x)}{x(n-p-2x-2)} \leqslant \frac{U_k(x)}{U_k(x+1)} \leqslant \frac{(n-x-1)(n+p-2x)}{x(n+p-2x-2)}.$$

Multiplying these inequalities and using initial values given by Lemma 4, we obtain the first claim.

The result for $V_k(x)$ follows in a similar manner with $g = \frac{x+1}{n-x-1}$. ∎

THEOREM 11. *Let* $0 \leqslant x < n/2$ *be an integer, then*

$$(K_k(x))^2 \leqslant \frac{x!\,(n-x)!}{\left\lfloor \frac{k}{2} \right\rfloor !^2 \left\lfloor \frac{n-k}{2} \right\rfloor !^2} \tau(n, k, x),$$

where $\tau(n, k, x)$ *is*

$$\frac{p^2+2n}{4(n-x)}, \qquad n, k \text{ even};$$

$$\frac{4}{n-x}, \qquad n \text{ even}, k \text{ odd};$$

$$\frac{2k+1}{n-x}, \qquad n \text{ odd}, k \text{ even};$$

$$\frac{2n-2k+1}{n-x}, \qquad n, k \text{ odd}.$$

Furthermore, if $\frac{n-p}{2} < x < \frac{n+p}{2}$, *then one can take* $\tau(n, k, x)$ *equals*

$$\frac{p}{p+2x-n}, \qquad n, k \text{ even};$$

$$\frac{16}{p+2x-n}, \qquad n \text{ even}, k \text{ odd};$$

$$\frac{8k}{(p+1)(p+2x-n)}, \qquad n \text{ odd}, k \text{ even};$$

$$\frac{8(n-k)}{(p+1)(p+2x-n)}, \qquad n, k \text{ odd}.$$

Proof.

$$(K_k(x))^2 \leqslant (K_k(x))^2 + xU_k(x) = V_k(x)$$

giving the first set of inequalities. The second follows from

$$\left(1-\frac{(n-2k)^2}{4x(n-x)}\right)v^2 \leqslant \left(1-\frac{(n-2k)^2}{4x(n-x)}\right)v^2 + \left(u\sqrt{\frac{n-x}{x}} - \frac{n-2k}{2\sqrt{x(n-x)}}\,v\right)^2 = U_k(x). \quad \blacksquare$$

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